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Modeling Heavy-Tailed Time Series

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For given $\ell \geq 1$, this value equals

$$
\begin{aligned}
& \Big[E(1-\mathrm{e}^{-f(Y_0)}) I_{\{\max_{j=1-l,\dots,-1}|Y_j|\le 1\}} + E\Big(\mathrm{e}^{-f(Y_0)} - \mathrm{e}^{-\sum_{j=0}^{1} f(Y_j)}\Big) I_{\{\max_{j=2-l,\dots,-1}|Y_j|\le 1\}} \\
& + \cdots + E\Big(\mathrm{e}^{-\sum_{j=0}^{\ell-1} f(Y_j)} - \mathrm{e}^{-\sum_{j=0}^{\ell} f(Y_j)}\Big) I_{\{\max_{j=\ell+1-l,\dots,-1}|Y_j|\le 1\}}\Big] \\
& + \Big[E\Big(\mathrm{e}^{-\sum_{j=0}^{\ell} f(Y_j)} - \mathrm{e}^{-\sum_{j=0}^{\ell+1} f(Y_j)}\Big) I_{\{\max_{j=\ell+2-l,\dots,-1}|Y_j|\le 1\}} \\
& + \cdots + E\Big(\mathrm{e}^{-\sum_{j=0}^{l-2} f(Y_j)} - \mathrm{e}^{-\sum_{j=0}^{l-1} f(Y_j)}\Big)\Big] \\
= \;& I_\ell^{(1)} + I_\ell^{(2)} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{\ell\to\infty}\lim_{l\to\infty} I_\ell^{(1)} &= \lim_{\ell\to\infty} E\Big(1-\mathrm{e}^{-\sum_{j=0}^{\ell} f(Y_j)}\Big) I_{\{\max_{j\le -1}|Y_j|\le 1\}} \\
&= E\Big(1-\mathrm{e}^{-\sum_{j=0}^{\infty} f(Y_j)}\Big) I_{\{\max_{j\le -1}|Y_j|\le 1\}} ,
\end{aligned}
$$

while

$$
\begin{aligned}
& \lim_{\ell\to\infty}\limsup_{l\to\infty} I_\ell^{(2)} \\
\le \;& \limsup_{l\to\infty} \Big[E\Big(\mathrm{e}^{-\sum_{j=0}^{\ell} f(Y_j)} - \mathrm{e}^{-\sum_{j=0}^{\ell+1} f(Y_j)}\Big) + \cdots + E\Big(\mathrm{e}^{-\sum_{j=0}^{l-2} f(Y_j)} - \mathrm{e}^{-\sum_{j=0}^{l-1} f(Y_j)}\Big)\Big] \\
= \;& \lim_{\ell\to\infty} E\mathrm{e}^{-\sum_{j=0}^{\ell} f(Y_j)} - E\mathrm{e}^{-\sum_{j=0}^{\infty} f(Y_j)} = 0 .
\end{aligned}
$$

$\square$

8. Max-stable processes with Fréchet marginals

Max-stable processes and random fields have recently attracted some attention for modeling spatio-temporal extremal phenomena. We give a short overview of results on the topic with special emphasis on max-stable time series.

Recall from Section 2.2 that max-stable distributions are the only non-degenerate limit distributions of (normalized and centered) partial maxima of an iid sequence. In particular, an iid sequence (X_t) with a max-stable distribution satisfies (2.7), i.e.,

$$
c_n^{-1}\big(\max(X_1,\dots,X_n) - b_n\big) \stackrel{d}{=} X , \quad n \ge 1 ,
$$

for suitable constants $c_n > 0$ and $d_n \in \mathbb{R}$. Here we will assume without loss of generality that X has a Fréchet distribution function $\Phi_\alpha(x) = \mathrm{e}^{-x^{-\alpha}}$, $x > 0$, and then $c_n = n^{1/\alpha}$ and $d_n = 0$.

A Fréchet random variable has the following representation which will be useful.

Lemma 8.1. *Let $0 < \Gamma_1 < \Gamma_2 < \cdots$ be an enumeration of the points of a unit rate homogeneous Poisson process on $(0,\infty)$ independent of an iid sequence (V_i) of positive random variables with $EV^\alpha < \infty$ for some $\alpha > 0$. Then $\sup_{i\ge 1} \Gamma_i^{-1/\alpha} V_i$ has a Fréchet $\Phi_\alpha^{EV^\alpha}$ distribution.*

Proof. Write $N(t) = \#\{i \ge 1 : \Gamma_i \le t\}$, $t \ge 0$, for the unit rate Poisson process on $(0,\infty)$. Let (U_t) be an iid sequence of random variables with a uniform distribution on $(0,1)$, independent of N and

(V_t). We notice that for $x>0$, using the order statistics property of N,

$$\begin{aligned} P\Big(\sup_{i\ge1}\Gamma_i^{-1/\alpha}V_i\le x\Big) &= \lim_{t\to\infty} E\Big[P\Big(\sup_{i\ge1}\Gamma_i^{-1/\alpha}V_i\le x\mid N(t)\Big)\Big]\\ &= \lim_{t\to\infty} E\Big[P\Big(\sup_{i\le N(t)}(tU_i)^{-1/\alpha}V_i\le x\mid N(t)\Big)\Big]\\ &= \lim_{t\to\infty} E\Big[P^{N(t)}\Big((tU_1)^{-1/\alpha}V_1\le x\Big)\Big]\\ &= \lim_{t\to\infty} \mathrm{e}^{-t\,P(V_1^\alpha>x^\alpha tU_1)}\\ &= \lim_{t\to\infty} \mathrm{e}^{-x^{-\alpha}\int_0^{tx^\alpha}P(V_1^\alpha>y)\,dy}\\ &= \mathrm{e}^{-x^{-\alpha}EV^\alpha}=\Phi_\alpha^{EV^\alpha}(x)\,. \end{aligned}\tag{8.1}$$

□

In what follows, we will consider extensions of the concept of max-stable distributions to the multivariate case. De Haan [59] introduced the notion of a (positive) *max-stable process* $(Y_t)_{t\in T}$, $T\subset\mathbb{R}$, by requiring that for iid copies $(Y_t^{(i)})_{t\in T}$, $i=1,2,\ldots$, of $(Y_t)_{t\in T}$,

$$n^{-1/\alpha}(\max_{i=1,\ldots,n}Y_t^{(i)})_{t\in T}\stackrel{d}{=}(Y_t)_{t\in T}\,,\quad n\ge1\,.\tag{8.2}$$

Then, in particular, all one-dimensional marginals of the process $(Y_t)_{t\in T}$ are Fréchet distributed, i.e. Y_t has distribution $\Phi_\alpha^{c(t)}$ for some function $c(t)\ge0$, $t\in T$.

Example 8.2. We consider an example from de Haan [59], p. 1195. Consider a unit rate homogeneous Poisson process on $(0,\infty)$ with points $\Gamma_1<\Gamma_2<\cdots$ independent of an iid sequence (U_i) with a uniform marginal distribution on $(0,1)$. Then $\sum_{i=1}^\infty\varepsilon_{(\Gamma_i^{-1/\alpha},U_i)}$ constitutes PRM$(\mu_\alpha\times\mathbb{LEB})$ on $(0,\infty)\times(0,1)$ and $\mu_\alpha(x,\infty)=x^{-\alpha}$, $x>0$. Let $(f_t)_{t\in T}$ be non-negative measurable functions on $(0,1)$ such that $Ef_t^\alpha(U)<\infty$.

We consider the process

$$Y_t=\sup_{i\ge1}\Gamma_i^{-1/\alpha}f_t(U_i)\,,\quad t\in T\,,$$

and we will show that it is a max-stable process. In view of the defining property (8.2) it suffices to show that for any distinct $t_i\in T$, $i=1,\ldots,m$, $m\ge1$, any $x_i>0$, $i=1,\ldots,m$, and $k\ge1$,

$$P\big(Y_{t_1}\le x_1,\ldots,Y_{t_m}\le x_m\big) = P^k\big(Y_{t_1}\le x_1k^{1/\alpha},\ldots,Y_{t_m}\le x_mk^{1/\alpha}\big)\,.\tag{8.3}$$

We notice that

$$P\big(Y_{t_1}\le x_1,\ldots,Y_{t_m}\le x_m\big)=P\Big(\sup_{i\ge1}\Gamma_i^{-1/\alpha}\max_{1\le j\le m}(f_{t_j}(U_i)/x_j)\le1\Big)\,.$$

An application of (8.1) yields

$$\begin{aligned} P\big(Y_{t_1}\le x_1,\ldots,Y_{t_m}\le x_m\big) &= \mathrm{e}^{-E\max_{1\le j\le m}(f_{t_j}(U)/x_j)^\alpha}\\ &= \mathrm{e}^{-\int_0^1\max_{1\le j\le m}(f_{t_j}(u)/x_j)^\alpha\,du}\,. \end{aligned}$$

Then (8.3) is straightforward.

This example already yields an almost complete characterization of the finite-dimensional distributions of a max-stable process. De Haan [59] proved the following result.

Theorem 8.3. *The finite-dimensional distributions of a max-stable sequence* $(Y_t)_{t\in\mathbb{N}}$ *with Fréchet marginals with index* $\alpha > 0$ *satisfy the relation*

$$P(Y_1 \le x_1, \dots, Y_m \le x_m) = \mathrm{e}^{-\int_{\mathbb{R}_+^m} \max_{t\le m}(y_t/x_t)^\alpha G_m(dy)}, \quad x_i > 0, \quad i = 1, \dots, m, \quad m \ge 1.$$

where G_m *is the* m*-dimensional restriction to* $\mathbb{R}_+^m$ *of a finite measure on* $\mathbb{R}_+^\infty$*. Moreover, there exists a finite measure* ρ *on* $[0,1]$ *such that* (Y_t) *has representation*

$$Y_t = \sup_{i\ge 1} \Gamma_i^{-1/\alpha} f_t(T_i), \quad t \in \mathbb{N},$$

where $((\Gamma_i^{-1/\alpha}, T_i))_{i=1,2,\dots}$ *is an enumeration of* $\mathrm{PRM}(\mu_\alpha \times \rho)$ *on* $(0,\infty) \times [0,1]$*,* (f_t) *are suitable non-negative measurable functions on* $[0,1]$ *such that* $Ef_t^\alpha(T_1) = \int_0^1 f_t^\alpha(x)\rho(dx) < \infty$.

De Haan [59] proved a similar result in the case $T = \mathbb{R}$ under the additional assumption that $(Y_t)_{t\in\mathbb{Z}}$ has stochastically continuous sample paths. Kabluchko [70] proved that any max-stable process $(Y_t)_{t\in T}$, $T \subset \mathbb{R}$, with Fréchet marginals of index $\alpha > 0$ has representation (on a sufficiently rich probability space)

$$Y_t = \sup_{i\ge 1} \Gamma_i^{-1/\alpha} f_t(T_i), \quad t \in T, \tag{8.4}$$

where $(f_t)_{t\in T}$ is a family of non-negative functions in $L^\alpha(\mathbb{E}, \mathcal{E}, \nu)$ and ν is a σ-finite measure on the Borel σ-field $\mathcal{E}$ of the state space $\mathbb{E}$, $\sum_{i=1}^\infty \varepsilon_{(\Gamma_i, T_i)}$ are the points of a $\mathrm{PRM}(\mathbb{LEB} \times \nu)$ on the state space $\mathbb{R}_+ \times \mathbb{E}$.

Using the same notation, one can introduce de Haan's [59] *extremal integral*

$$\int_{\mathbb{E}}^{\vee} f dM_\nu^\alpha = \sup_{i\ge 1} \Gamma_i^{-1/\alpha} f(T_i), \tag{8.5}$$

where, as above f is a non-negative function in $L^\alpha(\mathbb{E}, \mathcal{E}, \nu)$, and M_ν^α is an α*-Fréchet random sup-measure with control measure* ν. Stoev [114] proved that $\int_{\mathbb{E}}^{\vee} f dM_\nu^\alpha$ has various properties similar to the α-stable integrals; see Samorodnitsky and Taqqu [111]. A proof similar to the one in Example 8.2 yields that

$$\begin{aligned} P\Big(\int_{\mathbb{E}}^{\vee} f dM_\nu^\alpha \le x\Big) &= \exp\Big\{-x^{-\alpha} \int_{\mathbb{E}} f^\alpha d\nu\Big\} \\ &= \Phi_\alpha^{\int_{\mathbb{E}} f^\alpha d\nu}(x). \end{aligned}$$

The integral representation of a max-stable process is convenient. For example, for any $f_t \in L^\alpha(\mathbb{E}, \mathcal{E}, \nu)$, $x_t > 0$, $t = 1, \dots, m$, $m \ge 1$,

$$\begin{aligned} P\Big(\int_{\mathbb{E}}^{\vee} f_t dM_\nu^\alpha \le x_t, t = 1, \dots, m\Big) &= P\Big(\int_{\mathbb{E}}^{\vee} \max_{t=1,\dots,m} (f_t/x_t) dM_\nu^\alpha \le 1\Big) \\ &= \exp\Big\{-\int_{\mathbb{E}} \max_{t=1,\dots,m} (f_t/x_t)^\alpha d\nu\Big\}. \end{aligned}$$

We also have for $\mathbf{x} = (x_1, \dots, x_m) > \mathbf{0}$ and $y \to \infty$,

$$\begin{aligned} y\big[1 - P\Big(\int_{\mathbb{E}}^{\vee} f_t dM_\nu^\alpha \le y^{1/\alpha} x_t, t = 1, \dots, m\Big)\big] &= y\, P\Big(y^{-1/\alpha}\Big(\int_{\mathbb{E}}^{\vee} f_t dM_\nu^\alpha\Big)_{t=1,\dots,m} \notin [\mathbf{0}, \mathbf{x}]\Big) \\ &= y\Big(1 - \exp\Big\{-y^{-1} \int_{\mathbb{E}} \max_{t=1,\dots,m} (f_t/x_t)^\alpha d\nu\Big\}\Big) \\ &\to \int_{\mathbb{E}} \max_{t=1,\dots,m} (f_t/x_t)^\alpha d\nu = \mu_{m,\alpha}([\mathbf{0}, \mathbf{x}]^{\mathbf{c}}). \end{aligned} \tag{8.6}$$

Thus the finite-dimensional distributions of a max-stable process $(Y_t)_{t\in T}$ are regularly varying with index α and limiting measure $\mu_{m,\alpha}$ given by (8.6).

Recently, strictly stationary max-stable processes $(Y_t)_{t\in T}$ for $T=\mathbb{Z}$ or $T=\mathbb{R}$ have attracted some attention. Such a process has again integral representation

$$Y_t = \int_{\mathbb{E}}^{\vee} f_t dM_\nu^\alpha\,,\quad t\in T\,, \tag{8.7}$$

where the family of functions (f_t) has to satisfy some particular conditions to ensure strict stationarity, ergodicity, mixing, and other desirable properties; we refer to Kabluchko [70] and Stoev [114] for details.

Example 8.4. Assume that the strictly stationary max-stable process $(Y_t)_{t\in\mathbb{Z}}$ has representation (8.7). Since (Y_t) is regularly varying with index α can define its extremogram. For example, the extremogram with respect to the set $(1,\infty)$ is given by

$$\begin{aligned}\rho(h) &= \lim_{x\to\infty} P(x^{-1}Y_h>1 \mid x^{-1}Y_0>1)\\ &= \frac{P(x^{-1}\min(Y_0,Y_h)>1)}{P(Y_0>x)}\\ &= \lim_{x\to\infty}\frac{1-\exp\big\{-x^{-\alpha}\int_{\mathbb{E}}\min(f_0^\alpha,f_h^\alpha)d\nu\big\}}{1-\exp\big\{-x^{-\alpha}\int_{\mathbb{E}}f_0^\alpha d\nu\big\}}\\ &= \frac{\int_{\mathbb{E}}\min(f_0^\alpha,f_h^\alpha)d\nu}{\int_{\mathbb{E}}f_0^\alpha d\nu}\,.\end{aligned} \tag{8.8}$$

It is also straightforward to calculate the extremal index of (Y_t) provided it exists. Indeed, assuming $P(Y_0>a_n)=1-\mathrm{e}^{-a_n^{-\alpha}\int_{\mathbb{E}}f_0^\alpha d\nu}\sim n^{-1}$, i.e. $a_n\sim n^{1/\alpha}\big(\int_{\mathbb{E}}f_0^\alpha d\nu\big)^{1/\alpha}$, we have for $x>0$,

$$\begin{aligned}P\Big(a_n^{-1}\max_{t=1,\dots,n}Y_t\le x\Big) &= \exp\Big\{-a_n^{-\alpha}x^{-\alpha}\int_{\mathbb{E}}\max_{t=1,\dots,n}f_t^\alpha d\nu\Big\}\\ &= \Big[\Phi_\alpha(x)\Big]^{n^{-1}\int_{\mathbb{E}}\max_{t=1,\dots,n}f_t^\alpha d\nu\Big/\int_{\mathbb{E}}f_0^\alpha d\nu(1+o(1))}\,.\end{aligned}$$

If the limit

$$\theta_Y=\lim_{n\to\infty}\frac{1}{n}\frac{\int_{\mathbb{E}}\max_{t=1,\dots,n}f_t^\alpha d\nu}{\int_{\mathbb{E}}f_0^\alpha d\nu}$$

exists it is the extremal index of (Y_t).

We consider two popular examples of max-stable processes.

Example 8.5. The *Brown-Resnick process* (see [18]) has representation

$$Y_t=\sup_{i\ge1}\Gamma_i^{-1/\alpha}\,\mathrm{e}^{W_i(t)-0.5\,\sigma^2(t)}\,,\quad t\in\mathbb{R}\,, \tag{8.9}$$

where (Γ_i) is an enumeration of the points of a unit rate homogeneous Poisson process on $(0,\infty)$ independent of the iid sequence (W_i) of sample continuous mean zero Gaussian processes on $\mathbb{R}$ with stationary increments and variance function σ^2. The max-stable process (8.9) is stationary (Theorem 2 in Kabluchko et al. [71]; in this paper the authors also consider the case of max-stable random fields, i.e. W is a mean zero Gaussian random field with stationary increments) and its distribution only depends on the variogram $V(h)=\mathrm{var}(W(t+h)-W(t))$, $t\in\mathbb{R},h\ge0$. It follows from Example 2.1 in Dombry and Eyi-Minko [39] that the functions (f_t) in representation (8.4) satisfy the condition

$$\int_{\mathbb{E}}\min(f_0^\alpha,f_h^\alpha)d\nu\le c\,\overline{\Phi}(0.5\sqrt{V(h)})\,, \tag{8.10}$$

where Φ is the standard normal distribution. For example, if W is standard Brownian motion, $V(h) = h$, $\overline{\Phi}(0.5\sqrt{h}) \sim c\,\mathrm{e}^{-h/8} h^{-0.5}$, as $h \to \infty$. Notice that the right-hand side of (8.10) yields an exponential bound for the extremogram $\rho(h)$ in (8.8). Results in Dombry and Eyi-Minko [39] also show that (Y_t) is strongly mixing with exponential rate α_h.

Recently, the Brown-Resnick process has attracted some attention for modeling spatio-temporal extremes; see [70, 71, 114, 97]. The processes (8.9) can be extended to random fields on $\mathbb{R}^d$. These fields found various applications for modeling spatio-temporal extremal effects; see Kabluchko et al. [71], Davis et al. [26], Davison et al. [37]. The paper Davis et al. [34] collects some of the recent references on max-stable processes.

As a matter of fact, the Brown-Resnick process cannot be simulated in a naive way by mimicing the formula (8.9) and replacing the supremum over an infinite index set by a finite one. For example, assume that W is standard Brownian motion. Then $(\mathrm{e}^{W(t)-0.5t})_{t\ge 0}$ is a martingale with expectation 1 for every t. On the other hand, by virtue of the law of the iterated logarithm , $\mathrm{e}^{W(t)-0.5t} \to 0$ a.s. exponentially fast as $t \to \infty$. For every finite m, $\sup_{1\le i\le m} \Gamma_i^{-1/\alpha}\,\mathrm{e}^{W_i(t)-0.5\,\sigma^2(t)} \to 0$ exponentially fast as $t \to \infty$. This fact turns the simulation of (Y_t) into a complicated problem; see Oesting et al. [97].

Using the approach of Lemma 8.1, it is not difficult to see that for $0 < t_1 < \cdots < t_m \le T$, $m \ge 1$, and fixed T,

$$P\Big(\max_{i=1,\dots,m} Y_{t_i} \le x\Big) = \exp\Big\{-x^{-\alpha} E \max_{i=1,\dots,m} \mathrm{e}^{\alpha(W(t_i)-\sigma^2(t_i))}\Big\}\,,$$

and using the continuity of the sample paths,

$$\begin{aligned} P\Big(T^{-1/\alpha} \max_{0\le t\le T} Y_t \le x\Big) &= \exp\Big\{-x^{-\alpha}\frac{1}{T} E \max_{0\le t\le T} \mathrm{e}^{\alpha(W(t)-\sigma^2(t))}\Big\} \\ &\to \mathrm{e}^{-x^{-\alpha} c_\alpha}\,, \quad x > 0\,, \end{aligned}$$

where

$$c_\alpha = \lim_{T\to\infty} \frac{1}{T} E \max_{0\le t\le T} \mathrm{e}^{\alpha(W(t)-\sigma^2(t))}$$

exists and is known as *Pickands's constant*; see Pickands [101].

Example 8.6. We consider de Haan and Pereira's [60] *max-moving process*

$$Y_t = \sup_{i\ge 1} \Gamma_i^{-1/\alpha} f(t - U_i)\,, \quad t \in \mathbb{R}\,, \tag{8.11}$$

where f is a continuous Lebesgue density on $\mathbb{R}$ such that $\int_{\mathbb{R}} \sup_{|h|\le 1} f(x+h)\,dx < \infty$ and $\sum_{i=1}^\infty \varepsilon_{(\Gamma_i, U_i)}$ are the points of a unit rate homogeneous Poisson random measure on $(0,\infty)\times\mathbb{R}$.

The resulting process (Y_t) is α-max-stable and stationary. According to Example 2.2 in Dombry and Eyi-Minko [39],

$$\int_{\mathbb{E}} \min(f_0^\alpha, f_h^\alpha)\,d\nu \le c \int_{\mathbb{R}} \min(f^\alpha(-x), f^\alpha(h-x))\,dx\,, \quad h \ge 0\,,$$

and the right-hand side is a bound for the strong mixing rate α_h as well as for the extremogram $\rho(h)$. For example, if f is the standard normal density, this implies that (α_h) decays to zero faster than exponentially, i.e. the memory in this sequence is very short.

9. Large deviations

In the previous sections we frequently made use of the *principle of a single large jump* for a regularly varying sequence (X_t), i.e. it is often possible to make a statement about the extremal behavior of a random structure if we know the behavior of its largest component.

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